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# The ionisation equation in a relativistic gas

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**Abstract.** By deriving the relativistic form of the ionisation equation for a perfect gas it is shown that the usual Saha equation is valid to 3% for temperatures below one hundred million Kelvin. Beyond  $10^9$  K, the regular Saha equation is seriously incorrect and a relativistic distribution function for electrons must be taken into account. Approximate forms are derived when only the electrons are relativistic (appropriate up to  $10^{12}$  K) and also for the ultrarelativistic case (temperatures greater than  $10^{15}$  K).

## 1. Introduction

The ionisation equation has achieved substantial progress by relating the main line of the spectral sequence and luminosity effects in stellar spectra to the stellar temperature as well as to the electronic pressure. Although this theory gives very good results in many actual astrophysical situations, the case where the prevailing conditions are relativistic deserves fuller study, especially with respect to the electron gas; this may have some interest in the study of compact sources (Jones and Hardee 1979, Gould 1981) and would define the limits of validity of the non-relativistic formula.

The ionisation equation has been derived by Saha (1920, 1921) from thermodynamical considerations, by Fowler (1923, 1955) with the aid of statistical mechanics and without using the entropy concept, and by Menzel (1933) in a simplified version using Boltzmann's law of distribution of atoms in different energy states. We derive here the relativistic generalisation of the Saha equation by using the concepts and methods developed by Synge (1957) in the study of a relativistic gas mixture and introducing the most probable state by means of the same considerations as those used in the non-relativistic case by Aller (1963).

We give in § 2 the derivation of the relativistic ionisation equation in the simple case of a gas consisting of neutral atoms in their lowest state, singly ionised atoms and electrons; we exhibit in § 3 the limiting cases of the formula so obtained and in concluding remarks (§ 4) we examine the limits of validity of the non-relativistic formula.

## 2. The ionisation equation

We consider the thermal equilibrium of a gas mixture of three components: neutral atoms A of proper mass  $m_1$  in their lowest state, free ions  $A^+$  of proper mass  $m_2$  and

electrons  $e$  of proper mass  $m_3$  in dissociative equilibrium according to

$$A^+ + e \rightleftharpoons A. \tag{2.1}$$

In a given phase of the gas there will be  $\nu_1$  neutral atoms,  $\nu_2$  free ions and  $\nu_3$  electrons subject to the conditions

$$X = \nu_1 + \nu_2 = \text{constant}, \quad Y = \nu_1 + \nu_3 = \text{constant}. \tag{2.2}$$

Let us denote by  $N_A$  ( $A = 1, 2, 3$ ) the distribution functions of the atoms, ions and electrons respectively. For a small target of three-volume  $dS$  normal to a unit time-like vector  $u^r$  ( $u^r u_r = -1$ ) we have the occupation numbers

$$\nu_{Ai} = N_A dS d\Omega_{Ai} \tag{2.3}$$

where  $d\Omega_{Ai}$  defines for the state  $i$  of each type of particle an infinitesimal cell of the momentum three-space orthogonal to  $u^r$ ; it is assumed to have the same content for all types of particles, so that we may compare their relative probabilities. If  $G_{Ai}$  denote the statistical weights of the gas components in the state  $i$ ,

$$G_{Ai} = g_A dS d\Omega_{Ai}/h^3 \tag{2.4}$$

where  $g_A$  is the individual weight of the particle  $A$  and  $h^3$  the unit phase volume (Joos 1959, Sommerfeld 1964), the number  $W$  of permutabilities is given by

$$W = \left( X! Y! / \prod_A \prod_i \nu_{Ai} \right) \prod_A \prod_i G_{Ai}^{\nu_{Ai}}. \tag{2.5}$$

The equilibrium state, i.e. the most probable state, corresponds to the maximum of  $W$  or of the entropy integral.  $G_{Ai}$  and  $\nu_{Ai}$  are assumed to be very large so that the Stirling approximation may be used. Thus, taking into account (2.2), we get

$$d \log W = -[\log(N_1 h^3/g_1) + 2] dN_1 - [\log(N_2 h^3/g_2) + 1] dN_2 - [\log(N_3 h^3/g_3) + 1] dN_3 = 0. \tag{2.6}$$

This equation must be satisfied along with three other constraints, in thermal equilibrium; two of them are obtained from (2.2) and (2.3), keeping the phase cells the same during the variation of  $N_A$ ; the third constraint comes out from the conservation of the total four-momentum. Varying  $N_A$  and introducing the six Lagrange multipliers  $\alpha, \beta$  and  $\xi^r$ , we have

$$N_A = C_A \exp(\xi^r P_r) \tag{2.7a}$$

where  $P_r$  is the four-momentum of a typical particle and

$$C_1 = (g_1/h^3) \exp(-2 + \alpha + \beta), \quad C_2 = (g_2/h^3) \exp(-1 + \alpha), \\ C_3 = (g_3/h^3) \exp(-1 + \beta). \tag{2.7b}$$

Defining the numerical flux vector

$$N_{A,r} = C_A \int P_r \exp(\xi \lambda^r P_r) d\omega_A \tag{2.8a}$$

where  $d\omega_A$  denotes the absolute two-content of the momentum three-cell on the pseudosphere of radius  $m_{Ac}$  and  $\lambda^r$  the unit time-like vector

$$\lambda^r = \xi^r/\xi, \quad \xi = |\xi_r \xi^r|^{1/2}, \tag{2.8b}$$

we find

$$C_A = (N_{A(0)}\xi)/[4\pi m_A^2 c^2 K_2(x_A)], \quad x_A = m_A c \xi, \quad (2.9a)$$

in which  $\lambda'$  determine the rest frame of the gas and  $N_{A(0)}$  the numerical rest densities of the components of the gas; the Bessel function  $K_n(x)$  is defined for our purposes by

$$K_n(x) = \int_0^\infty \exp(x \cosh t) \cosh nt \, dt. \quad (2.9b)$$

Combining (2.7b) and (2.9a) we get

$$I_R \stackrel{\text{def}}{=} \left( \frac{N_{2(0)} N_{3(0)}}{N_{1(0)}} \right)_R = \frac{4\pi}{\xi h^3} \frac{g_2 g_3}{g_1} \left( \frac{m_2 m_3 c}{m_1} \right)^2 \frac{K_2(x_2) K_2(x_3)}{K_2(x_1)}. \quad (2.10)$$

This equation constitutes the relativistic generalisation of the Saha ionisation equation for a perfect gas with the energy-momentum tensor

$$T_{rs} = \sum_A C_A \int P_r P_s \exp(\xi \lambda' P_r) \, d\omega_A \quad (2.11)$$

where the effective temperature  $T$  is given by

$$T = c/k\xi \quad (2.12)$$

with  $k$  Boltzmann's constant and  $c$  the velocity of light.

### 3. Limiting cases

We show first that (2.14) reduces to the Saha ionisation equation for low effective temperature

$$x_A = m_A c^2/kT \gg 1 \quad \text{for all components} \quad (3.1)$$

and then we investigate the cases when:

(a) equation (3.1) is satisfied only for atoms and ions whereas for electrons  $x_3 \ll 1$ ;

(b) equation (3.1) is replaced by  $x_A \ll 1$  for all components, i.e. for very high effective temperature.

#### 3.1. The non-relativistic case

When (3.1) is satisfied for all components ( $T \leq 10^6$  K) we may approximate  $K_2(x_A)$  by

$$K_2(x_A) \approx (\pi/2x_A)^{1/2} \exp(-x_A) \quad (3.2)$$

and equation (2.10) yields

$$I_{NR} \stackrel{\text{def}}{=} \left( \frac{N_{2(0)} N_{3(0)}}{N_{1(0)}} \right)_{NR} = \frac{g_2 g_3}{g_1 h^3} \left( \frac{2\pi m_2 m_3}{m_1} kT \right)^{3/2} \exp\left(-\frac{\chi}{kT}\right) \quad (3.3a)$$

where  $\chi$  is the ionisation energy such that

$$\chi = (m_2 + m_3 - m_1)c^2. \quad (3.3b)$$

If, moreover, we assume  $m_2/m_1 \approx 1$ , i.e. atom and ion masses comparable, we get the well known Saha ionisation equation

$$I_S \stackrel{\text{def}}{=} \left( \frac{N_{2(0)}N_{3(0)}}{N_{1(0)}} \right)_S = \frac{g_2g_3}{g_1h^3} (2\pi m_3 kT)^{3/2} \exp\left(-\frac{\chi}{kT}\right). \tag{3.4}$$

**3.2. The relativistic electrons case**

When the temperature lies between  $10^{10}$  K and  $10^{12}$  K, atoms and ions are non-relativistic whereas electrons are highly relativistic ( $x_3 \ll 1$ ) and

$$K_2(x_3) \approx 2/x_3^2. \tag{3.5}$$

We get, taking into account (3.5) and  $m_1/m_2 \approx 1$ ,

$$\left( \frac{N_{2(0)}N_{3(0)}}{N_{1(0)}} \right)_{R_e} \approx \frac{g_2g_3}{g_1h^3} 8\pi \left( \frac{kT}{c} \right)^3 \left( 1 + \frac{m_3c^2}{kT} + \dots \right) \exp\left(-\frac{\chi}{kT}\right). \tag{3.6}$$

**3.3. The ultrarelativistic case**

When the effective temperature is very high ( $T \geq 10^{15}$  K) all particles are highly relativistic ( $x_A \ll 1$ ). Assuming  $m_1/m_2 \approx 1$ , we get the ultrarelativistic ionisation formula

$$\left( \frac{N_{2(0)}N_{3(0)}}{N_{1(0)}} \right)_{UR} \approx \frac{g_2g_3}{g_1h^3} \left( \frac{kT}{c} \right)^3. \tag{3.7}$$

**4. Concluding remarks**

From (2.10), (3.3a) and (3.4) we obtain the useful formula

$$I_R/I_S \approx I_R/I_{NR} = L_2L_3/L_1 \tag{4.1}$$

where  $L_A = K_2(x_A)(2x_A/\pi)^{1/2} e^{x_A}$  with  $x_1 = 1.087 \times 10^{13} T^{-1}$ ,  $x_2 = 1.086 \times 10^{13} T^{-1}$  and  $x_3 = 5.929 \times 10^9 T^{-1}$ . However, for temperatures between  $10^8$  K and  $10^{10}$  K,  $L_1$  and  $L_2$  may be approximated to 1 so that such astrophysical situations include possibly cases where electrons are weakly relativistic. Using values of  $K_2(x)$  from Abramowitz and Stegun (1964), we may give as an illustration the following table.

<i>T</i>	$10^4$	$10^6$	$10^8$	$3.10^8$	$10^9$	$10^{10}$	$10^{12}$
$I_R/I_S$	1	1	1.03	1.09	1.34	5.76	$3.5 \times 10^3$

It appears therefore that:

- (a) the usual Saha equation can be safely used below  $10^6$  K;
- (b) it remains valid to 3% for temperatures about  $10^8$  K;
- (c) it is seriously incorrect beyond  $10^9$  K as the relativistic Jüttner distribution function should then be used for electrons.

Beyond  $10^{11}$  K we have  $x_3 \ll 0.1$  and  $x_3^2 K_2(x_3) \approx 2$  so that  $L_3 \approx 0.1105 (10^{-9} T)^{3/2}$ ; relativistic corrections are then very important, as the equilibrium electronic density

$N_3(0)$  is also  $L_3$  times higher than the corresponding non-relativistic density. However, the above considerations apply only to the case of a perfect gas where particles interact by binary collisions.

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